Statistical and algorithmic aspects of optimal quantization and sparse Wasserstein barycenters

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Example 1: Optimal quantization

μ (Target Measure)

ν

$\frac{1}{N}\sum_{k=1}^{N}\delta_{y_k}$

$\sum_{k=1}^{N}\pi_k\delta_{y_k}$

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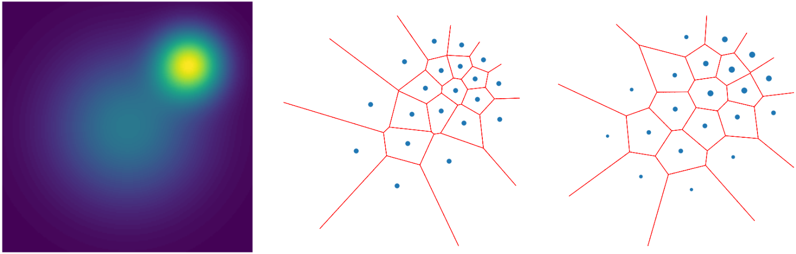

Figure: Optimal and uniform quantization of a Gaussian mixture

Simulations made using the library PyMongeAmpere by Quentin Mérigot.

Example 2: Empirical sparse Wasserstein barycenter

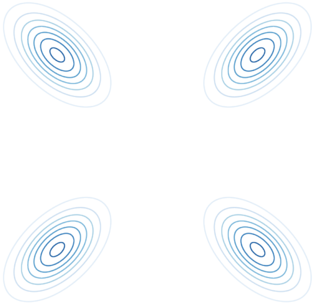

Figure: Wasserstein barycenter and its sample average approximation

Simulations made using the POT Library.

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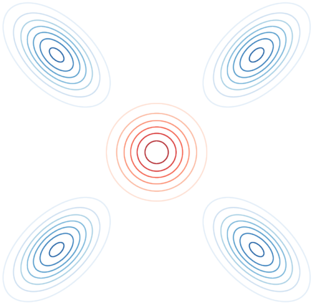

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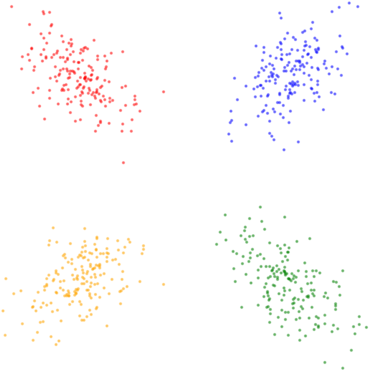

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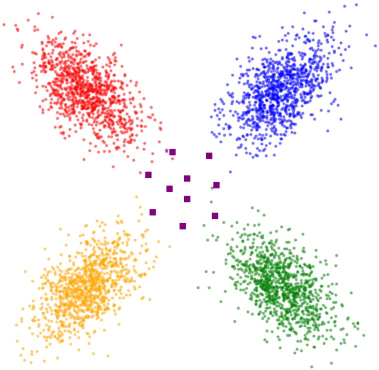

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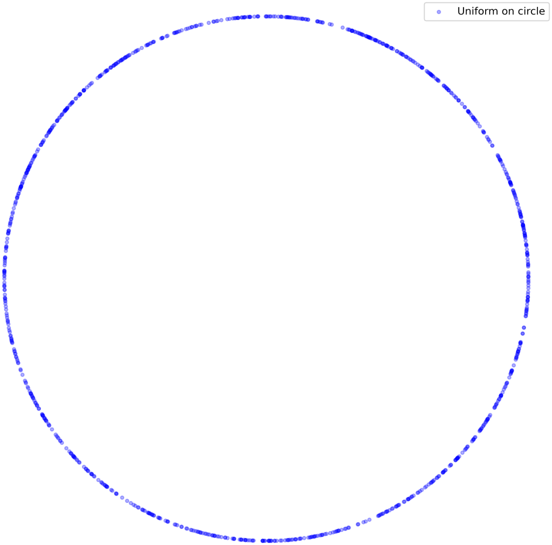
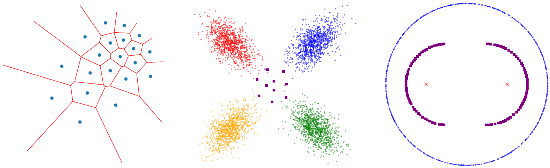
Example 3: Wasserstein geodesics



Connection between these examples



What do all these examples have in common?

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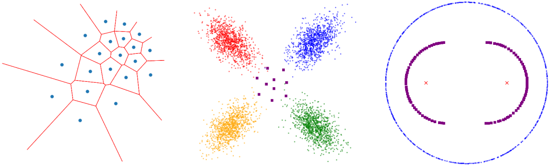

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They are all instances of problems of optimization of point clouds (seen as **probability measures**) and/or their probability weights.

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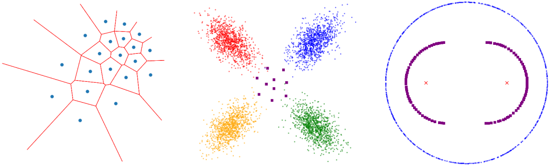

What do all these examples have in common?

They are all instances of problems of optimization of point clouds (seen as **probability measures**) and/or their probability weights. More specifically, theses examples aim at solving

$$\min_{(Y,\pi)\in A} \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{D}\left(\mu^\ell, \sum_{i=1}^{N} \pi_i \delta_{y_i}\right)$$

where $\mu^1, \ldots, \mu^L$ are a collection of probability measures, A is a closed, non-empty subset of $(\mathbb{R}^d)^N \times \Delta_N$ and and $\mathbb{D}$ is a divergence between measures.

Objectives

- Define a good notion of metric between arbitrary measures (**discrete** or **continuous**)

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- Study interesting sub cases of

$$\min_{(Y,\pi)\in A} \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{D}\left(\mu^{\ell}, \sum_{i=1}^{N} \pi_i \delta_{y_i}\right)$$

In particular

1. Algorithmic properties
2. Statistical properties.

Optimal transport

Optimal transport measures the minimal cost of transporting the target measure μ onto ν with couplings γ, according to a cost $c : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$.

$$\boxed{\mathrm{OT}_c(\mu, \nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\gamma(x, y).}$$

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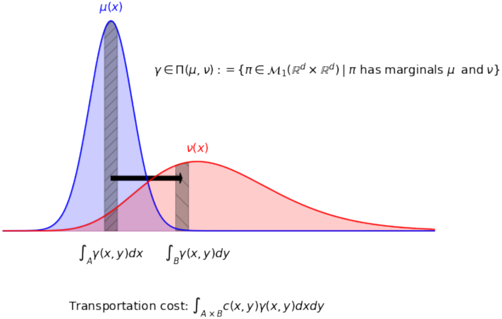

Dual formulation

Kantorovich duality:

$$\begin{aligned}\mathrm{OT}_c(\mu,\nu) &= \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y)d\gamma(x,y)\\ &= \sup_{\substack{\varphi,\psi\in\mathcal{C}(\mathbb{R}^d)\\ \varphi\oplus\psi\le c}} \int_{\mathbb{R}^d}\varphi d\mu + \int_{\mathbb{R}^d}\psi d\nu.\end{aligned}$$

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c-transform: $\varphi^c(x) := \inf_{y\in\mathbb{R}^d} c(x,y) - \psi(y)$.

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$$= \sup_{\substack{\varphi,\psi\in\mathcal{C}(\mathbb{R}^d)\\ \varphi\oplus\psi\leq c}} \int_{\mathbb{R}^d} \varphi d\mu + \int_{\mathbb{R}^d} \psi d\nu.$$

c-transform: $\varphi^c(x) := \inf_{y\in\mathbb{R}^d} c(x,y) - \psi(y)$. Constraint $\varphi \oplus \psi \leq c$ is thus equivalent to $\psi \leq \varphi^c$.

$$\mathrm{OT}_c(\mu,\nu) = \sup_{\psi\in\mathcal{C}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \varphi^c d\mu + \int_{\mathbb{R}^d} \varphi d\nu.$$

The quadratic cost

If $c(x,y) := \|x-y\|^p,\ p \geq 1$ then $W_p(\mu,\nu) := \mathrm{OT}_c(\mu,\nu)^{1/p}$ is a **distance** over the probability space $\mathcal{M}_1(\mathbb{R}^d)$ called the **Wasserstein distance**.

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Moreover, if μ has density, then by Brenier's Theorem there exists a mapping $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned}\mathrm{OT}_c(\mu, \nu) &= W_p^p(\mu, \nu) \\ &= \int_{\mathbb{R}^d} \|x - T(x)\|^p d\mu(x)\end{aligned}$$

T is called the **optimal transport map**.

The semi discrete setting

Suppose $\nu := \sum_{i=1}^{N} \pi_i \delta_{y_i},\ Y \in (\mathbb{R}^d)^N,\ \pi \in \Delta_N$.

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\mathrm{OT}_c(\mu, \nu) &= \sup_{\varphi \in \mathcal{C}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \varphi^c d\mu + \int_{\mathbb{R}^d} \varphi d\nu \\
&= \sup_{\varphi \in \mathbb{R}^N} \int_{\mathbb{R}^d} \min_{i=1,\dots,N} c(x, y_i) - \varphi_i \, d\mu(x) + \sum_{i=1}^{N} \pi_i \varphi_i \\
&= \sum_{i=1}^{N} \int_{L_i^c(Y, \varphi^*)} c(x, y_i) - \varphi_i^* \, d\mu(x) + \sum_{i=1}^{N} \pi_i \varphi_i^*
\end{aligned}
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\end{aligned}$$

where

$$L_i^c(Y, \varphi^*) = \left\{ x \in \mathbb{R}^d \mid \, c(x, y_i) - \varphi_i^* \leq c(x, y_j) - \varphi_j^*, \, j \neq i \right\}.$$

The semi discrete setting

If $c(x, y) = \|x - y\|^p$, the power cells form a convex tesselation of $\mathbb{R}^d$ and define the OT map $T = \sum_{i=1}^N y_i 1_{L_i(Y,\varphi^*)}$.

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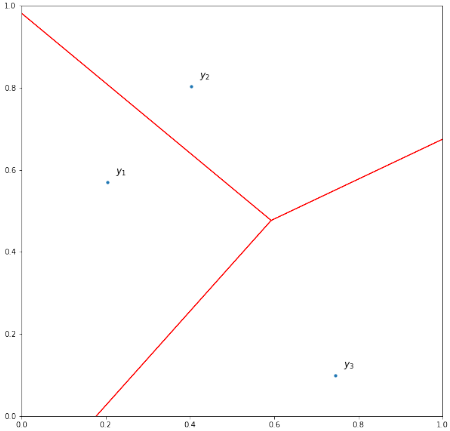

Figure: Example of semi-discrete OT between $\mu \sim \mathcal{U}_{[0,1]^2}$ and $\frac{1}{3}(\delta_{y_1} + \delta_{y_2} + \delta_{y_3})$.

Entropic Optimal transport

$$W_{\epsilon,p}(\mu,\nu) = \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\|^p d\gamma(x,y) + \epsilon \mathrm{KL}(\gamma, \mu \bigotimes \nu)$$

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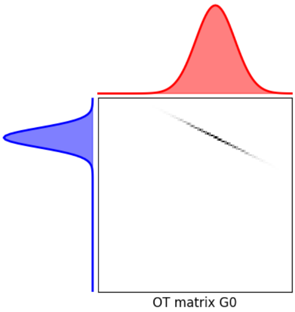

Figure: *POT library example of a $W_{\epsilon,2}$-transport plan*

Wider because it penalizes couplings too far from the product measure

Sliced optimal transport

1-D optimal transport: if $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$:

$$W_p^p(\mu,\nu) = \int_0^1 \mid F_\mu^{-1}(t) - F_\nu^{-1}(t) \mid^p dt$$

where F_α^{-1} denotes the generalized inverse of the c.d.f of α.

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The sliced and max-sliced Wasserstein distances leverage the closed form of 1-d OT through projections on the line with directions θ.

$$SW_p^p(\mu, \nu) = \int_{\mathbb{S}^{d-1}} W_p^p(P_\theta \sharp \mu, P_\theta \sharp \nu) d\sigma(\theta), \ \sigma \sim \mathcal{U}_{\mathbb{S}^{d-1}}$$

$$\text{max-}SW_p^p(\mu, \nu) = \max_{\theta \in \mathbb{S}^{d-1}} W_p^p(P_\theta \sharp \mu, P_\theta \sharp \nu).$$

Our object of study

Let $\mu^1, \ldots, \mu^L \in \mathcal{M}_1(\mathbb{R}^d)$, we study

$$\min_{(Y,\pi)\in A} \mathsf{F}_{\mathbb{D}}\left(\mu^1, \ldots, \mu^L, \sum_{i=1}^{N} \pi_i \delta_{y_i}\right) := \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{D}\left(\mu^\ell, \sum_{i=1}^{N} \pi_i \delta_{y_i}\right).$$

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A	$L = 1$	$L > 1$
$(\mathbb{R}^d)^N \times \Delta_N$	Optimal quantization	OT barycenter with sparse support
$(\mathbb{R}^d)^N \times \{\bar{\pi}\}$	Constrained quantization	Free support OT barycenter
$\{\bar{Y}\} \times \Delta_N$	-	Fixed support OT barycenter

The objective function

Focus on $\mathbb{D} = W_p^p$ and $L = 1$

$$\boxed{(Y, \pi) \mapsto \sup_{w \in \mathbb{R}^N} \int_{\mathbb{R}^d} \|x - y_i\|^p - w_i d\mu(x) + \sum_{i=1}^{N} w_i \pi_i}$$

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- Non convex in Y.
- GD works in practice but convergence guarantees are hard.

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In this section we provide such guarantees.

Uniform and optimal quantization

$$\min_{Y:=(y_1,\dots,y_N)} \frac{1}{2} W_2^2 \left(\mu, \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i} \right) = \min_{Y:=(y_1,\dots,y_N)} F_N(Y)$$

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where
$F_N(Y) = \max_{w \in \mathbb{R}^N} \frac{1}{2} \left\{ \int_{\mathbb{R}^d} \left(\min_{i=1,\dots,N} \|x - y_i\|^2 - w_i \right) f(x) dx + \frac{1}{N} \sum_{i=1}^{N} w_i \right\}$

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(Uniform Quantization)

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(Uniform Quantization)

$$\boxed{\min_{Y, \pi > 0, \sum_{i=1}^{N} \pi_i = 1} \frac{1}{2} W_2^2 (\mu, \sum_{i=1}^{N} \pi_i \delta_{y_i}) = \min_{Y:=(y_1,\ldots,y_N)} G_N(Y)}$$

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(Optimal Quantization)

Voronoï and Laguerre tesselation

- i'th Voronoï cell:
  $V_i(Y) = \{x \in \mathbb{R}^d \mid \|x - y_i\|^2 < \|x - y_j\|^2 \ \forall j = 1, ..., N\}$

Optimal Quantization

Uniform Quantization

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- i'th Laguerre cell:
  $L_i(Y, w) = \{x \in \mathbb{R}^d \mid \|x - y_i\|^2 - w_i < \|x - y_j\|^2 - w_j \; \forall j = 1, ..., N\}$

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Optimal Quantization

$$\begin{aligned} G_N(Y) &= \frac{1}{2} \int_{\mathbb{R}^d} \min_{i=1,\dots,N} \left\| x - y_i \right\|^2 f(x) dx \\ &= \frac{1}{2} \sum_{i=1}^{N} \int_{V_i(Y)} \left\| x - y_i \right\|^2 f(x) dx \end{aligned}$$

Discrete
Uniform Quantization

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Optimal Quantization

$$G_N(Y) = \frac{1}{2} \int_{\mathbb{R}^d} \min_{i=1,\dots,N} \|x - y_i\|^2 f(x)dx$$
$$= \frac{1}{2} \sum_{i=1}^{N} \int_{V_i(Y)} \|x - y_i\|^2 f(x)dx$$

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$$F_N(Y) = \max_{w \in \mathbb{R}^N} \frac{1}{2} \left\{ \int_{\mathbb{R}^d} \left(\min_{i=1,\dots,N} \|x - y_i\|^2 - w_i \right) f(x)dx + \frac{1}{N} \sum_{i=1}^{N} w_i \right\}$$
$$= \max_{w \in \mathbb{R}^N} \frac{1}{2} \left\{ \sum_{i=1}^{N} \int_{L_i(Y,w)} \left(\|x - y_i\|^2 - w_i \right) f(x)dx + \frac{1}{N} \sum_{i=1}^{N} w_i \right\}$$

Lloyd's algorithm[1]: two equivalent formulations

Can be described in two ways

1. Choose an initial points cloud Y.

[1] Lloyd, Stuart P. Least squares quantization in PCM, 1982.

Lloyd's algorithm[1]: two equivalent formulations

Can be described in two ways

1. Choose an initial points cloud Y.
2. Construct either a Voronoï (OQ) or a Laguerre tessellation (UQ) of the support of μ.

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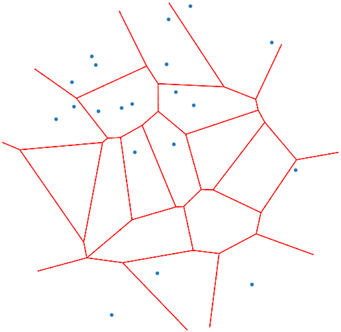
Lloyd's algorithm for (Uniform Quantization): an example with $\mu \sim \mathcal{N}_{\mathbb{S}^1}(0_2, \sigma^2 I_2)$



Simulations made using the library PyMongeAmpere by Quentin Mérigot.

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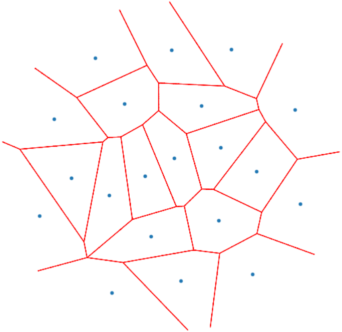

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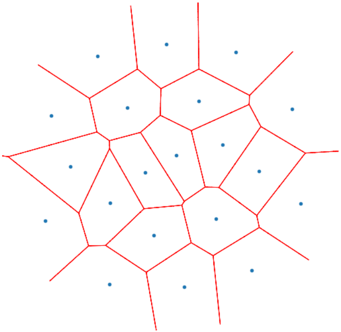

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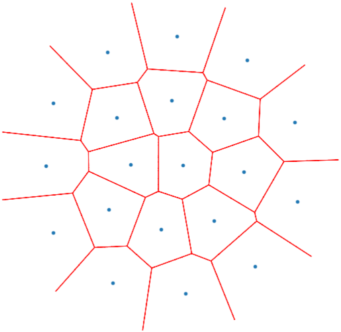

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Convergence of Lloyd's algorithm for OQ and UQ

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$F_N(Y) = \max_{w \in \mathbb{R}^N} \frac{1}{2} \left\{ \int_{\mathbb{R}^d} \left(\min_{i=1,\ldots,N} \|x - y_i\|^2 - w_i \right) f(x) dx + \frac{1}{N} \sum_{i=1}^{N} w_i \right\}$

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Question:

What about $(Y_n)_{n \geq 1}$ and $(Z_n)_{n \geq 1}$?

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Main result

Theorem

Let μ be a continuous probability measure supported on a compact and convex subset of $\mathbb{R}^d$. Suppose its density is ???. Then the iterates of Lloyd for both uniform (UQ) and optimal quantization (OQ) converge.

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Remark

Convexity of Supp(μ) is not necessary for uniform quantization.

Convergence of gradient schemes in non-convex setting

- F_N and G_N both non convex.

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KL inequality

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f verifies a KL inequality at x^* if there exists a neighborhood U of x^*, $c > 0$, $\eta > 0$ and a strictly increasing positive function $\Psi : [0, \eta[\longrightarrow \mathbb{R}$ such that:

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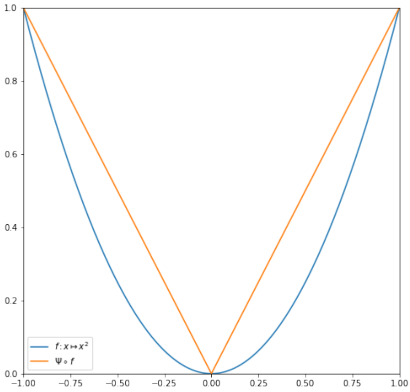
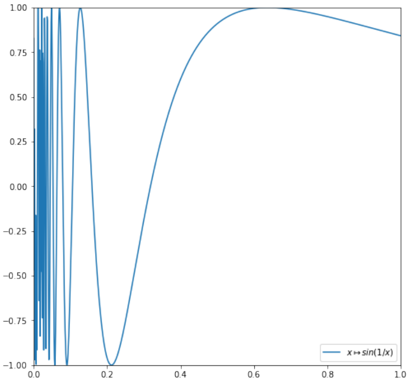
A function whose differential go to zero can have infinitely many oscillations as it goes to 0. The KL property prevents that from happening.

KL inequality

A case where the function is KL... and another one where it is not (at 0).





f:x$\mapsto x^2$, $\Psi : x \mapsto \sqrt{x}$ $\quad$ $g : x \mapsto \sin 1/x$.

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Question:

Why are F_N and G_N KL? Because they are definable in an o minimal structure.

A (very quick) overview on o-minimality

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Examples of globally subanalytic (GSA) functions

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Think of it as analytic functions restricted to a semi-algebraic compact subset

Proving the definability of G_N and F_N

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From optimization to statistics

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for $\mathbb{D} = W_p^p,\ W_{\epsilon,p},\ SW_p^p,\ \text{max-}SW_p^p$. More specifically we investigate the **sample average approximation** of $\mathsf{F}_{\mathbb{D}}$.

Sparse optimal transport barycenters

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Empirical risk minimization:

For a general loss function L

$$\mathbb{E}\left[L(f_n(X)) - \min_{f\in\mathcal{F}} \mathbb{E}[L(f(X))]\right] \leq 2\mathbb{E}\left[\sup_{f\in\mathcal{F}} \mid \mathbb{P}_n L(f) - \mathbb{E}[L(f(X))] \mid\right]$$
$$\lesssim \frac{\mathcal{C}(n)}{\sqrt{n}}$$

where f_n is a minimizer of $\mathbb{P}_n L(f)$ and $\mathcal{C}(n)$ is a measure of complexity (Rademacher, VC dimension, log-entropy, ...)

The K-means as an empirical risk minimization scheme

It was shown[10] that empirical optimal quantization (or K-means) behaves as

$$\mathbb{E}\left[D_2(\mu, Y_n) - \min_{Y \in \overline{\mathrm{B}_d(0,R)}^N} D_2(\mu, Y)\right] \lesssim \sqrt{\frac{N\log(N)}{n}}$$

where $D_2(\mu, Y) = \int_{\mathbb{R}^d} \min_{i=1,\ldots,N} \|x - y_i\|^2 d\mu(x)$.

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Question:

Does this result also hold for the more general problem

$$\min_{(Y,\pi)\in A} \frac{1}{L}\sum_{l=1}^{L} \mathbb{D}\left(\mu^l, \sum_{i=1}^{N} \pi_i \delta_{y_i}\right) \quad A \subset (\mathbb{R}^d)^N \times \Delta_N \text{ ?}$$

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$$\mathbb{E}\left[\mathsf{F}_{\mathbb{D}}\left(\mu^1, \ldots, \mu^L, \sum_{i=1}^N \pi_i^n \delta_{y_i^n}\right) - \min_{(Y,\pi)\in A} \mathsf{F}_{\mathbb{D}}\left(\mu^1, \ldots, \mu^L, \sum_{i=1}^N \pi_i \delta_{y_i}\right)\right]$$
$$\lesssim \sqrt{\frac{N}{n}}.$$

Tightness of the bound

Lemma (Bartlett et.al, 2002)

Let $\mathbb{D} = W_2^2$*,* $n \in \mathbb{N}^*$ *and* $R > 0$*. For any* $N \in \mathbb{N}^*$ *there exists* $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ *and* $\pi \in \Delta_N$ *such that*

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where $(y_1^n, \ldots, y_N^n)$ *denotes a minimizer of* $(y_1, \ldots, y_N) \mapsto \mathsf{F}_{W_2^2}\left(\mu_n^1, \ldots, \mu_n^L, \sum_{i=1}^N \pi_i \delta_{y_i}\right)$.

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- Our bound is min-max optimal in n.
- But not optimal in N (depends heavily on the target measures).

Illustration of the convergence

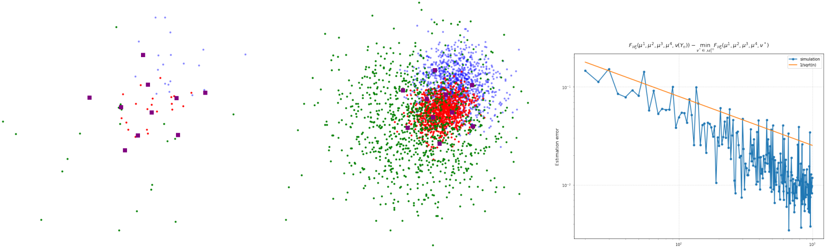

Figure: Empirical barycenter with increasing sample size.

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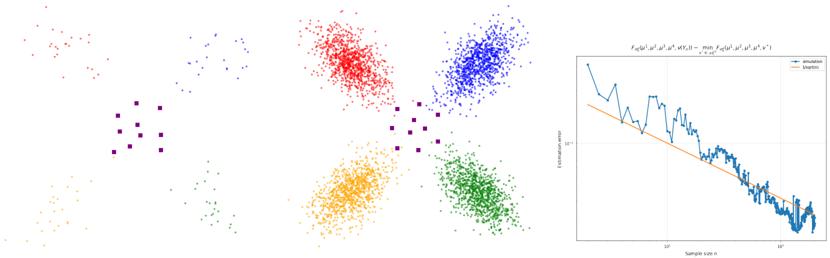

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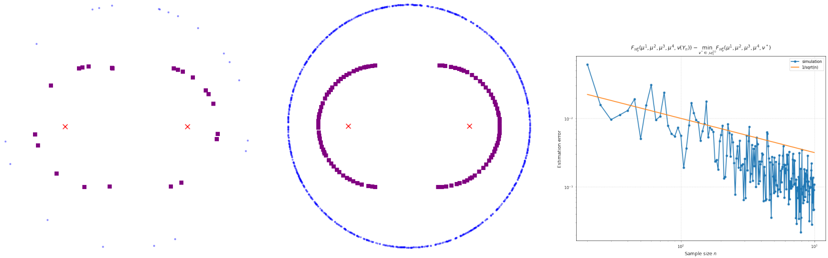

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Sketch of proof: generalization error

- Let $\nu_n^* := \sum_{i=1}^N \pi_i^n \delta_{y_i^n}$ be a minimizer of $\mathsf{F}_{\mathbb{D}}(\mu_n^1, \ldots, \mu_n^L, \nu(Y, \pi))$ over $A \subset (\mathbb{R}^d)^N \times \Delta_N$.
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(Generalization error)

Sketch of proof: control of the generalization error

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$$\leq \frac{1}{L} \sum_{\ell=1}^L \left| W_p^p\left(\mu_n^\ell, \sum_{i=1}^N \pi_i \delta_{y_i}\right) - W_p^p\left(\mu^\ell, \sum_{i=1}^N \pi_i \delta_{y_i}\right) \right|$$

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$$\leq \frac{1}{L} \sum_{\ell=1}^L \sup_{\|w\| \leq K} \left| \mathbb{P}_n^\ell f_{Y,w} - \mathbb{E}_{X^\ell \sim \mu^\ell}[f_{Y,w}(X^\ell)] \right|$$

Sketch of proof: Log-entropy

Taking the expectation on both sides

$$\mathbb{E}\left[\sup_{(Y,\pi)} \left| \mathsf{F}_{\mathbb{D}}\left(\mu_n^1, \ldots, \mu_n^L, \sum_{i=1}^{N} \pi_i \delta_{y_i}\right) - \mathsf{F}_{\mathbb{D}}\left(\mu^1, \ldots, \mu^L, \sum_{i=1}^{N} \pi_i \delta_{y_i}\right) \right|\right]$$

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$$\mathbb{E}\left[\sup_{(Y,\pi)}\left|\mathsf{F}_{\mathbb{D}}\left(\mu_n^1,\ldots,\mu_n^L,\sum_{i=1}^N \pi_i\delta_{y_i}\right)-\mathsf{F}_{\mathbb{D}}\left(\mu^1,\ldots,\mu^L,\sum_{i=1}^N \pi_i\delta_{y_i}\right)\right|\right]$$

$$\leq \frac{1}{L}\sum_{\ell=1}^L \mathbb{E}\left[\sup_{(Y,\pi,w)}\left|\frac{1}{n}\sum_{i=1}^n f_{Y,w}(X_i^\ell)-\mathbb{E}_{X^\ell\sim\mu^\ell}[f_{Y,w}(X^\ell)]\right|\right]$$

Sketch of proof: Log-entropy

Taking the expectation on both sides

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{(Y,\pi)}\left| \mathsf{F}_{\mathbb{D}}\left(\mu_n^1,\dots,\mu_n^L,\sum_{i=1}^{N}\pi_i\delta_{y_i}\right) - \mathsf{F}_{\mathbb{D}}\left(\mu^1,\dots,\mu^L,\sum_{i=1}^{N}\pi_i\delta_{y_i}\right)\right|\right] \\
&\leq \frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}\left[\sup_{(Y,\pi,w)}\left|\frac{1}{n}\sum_{i=1}^{n} f_{Y,w}(X_i^\ell) - \mathbb{E}_{X^\ell\sim\mu^\ell}[f_{Y,w}(X^\ell)]\right|\right] \\
&= \frac{1}{L}\sum_{\ell=1}^{L}\sup_{f\in\mathcal{F}_p}\left|\mathbb{P}_n^\ell(f) - \mathbb{E}_{X\sim\mu^\ell}[f(X)]\right|
\end{aligned}
$$

Sketch of proof: Log-entropy

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$$\mathbb{E}\left[\sup_{(Y,\pi)}\left|\mathsf{F}_{\mathbb{D}}\left(\mu_n^1,\dots,\mu_n^L,\sum_{i=1}^N \pi_i\delta_{y_i}\right)-\mathsf{F}_{\mathbb{D}}\left(\mu^1,\dots,\mu^L,\sum_{i=1}^N \pi_i\delta_{y_i}\right)\right|\right]$$

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$$=\frac{1}{L}\sum_{\ell=1}^L \sup_{f\in\mathcal{F}_p}\left|\mathbb{P}_n^\ell(f)-\mathbb{E}_{X\sim\mu^\ell}[f(X)]\right|$$

$$\leq \frac{C_1}{\sqrt{n}}\mathbb{E}\left[\int_0^{C_2}\sqrt{\log\Big(2\mathcal{N}(\tau/2,\mathcal{F}_p,\|\cdot\|_{\mathbb{L}^2(\mu_n^\ell)})\Big)}\,d\tau\right]$$

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$$\lesssim \sqrt{\frac{N}{n}}$$

Conclusion

Although non-convex, problems of points cloud optimization with **OT** losses have good rigidity properties under definability assumption on the target measures. This allows for good behavior of gradient sequences (such as **Lloyd's algorithms**).

Furthermore these problems have sample complexity $O(n^{-1/2})$, independent (exponentially) of the dimension. This rate is optimal in general.

Thank you for your attention!

Statistical aspect of optimal transport

Study of asymptotic behavior of $\mathbb{D}(\mu, \mu_n)$ and $\mathbb{D}(\mu, \nu_n)$ $\nu \neq \mu$.

[11] On the rate of convergence in Wasserstein distance of the empirical measure, N.Fournier and A.Guillin. 2013

Statistical aspect of optimal transport

Study of asymptotic behavior of $\mathbb{D}(\mu, \mu_n)$ and $\mathbb{D}(\mu, \nu_n)$ $\nu \neq \mu$. In all generality it depends on dimensionality. For instance if μ's support is compact[11]:

$$\mathbb{E}\left[W_p(\mu_n, \mu)\right] \lesssim \begin{cases} n^{-1/d} & \text{if } d > 2p \\ n^{-1/2p} \log(n)^{1/p} & \text{if } d = 2p \\ n^{-1/2p} & \text{if } d < 2p \end{cases}$$

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and by triangle inequality this is also true for for $\mid W_p(\mu_n, \nu) - W_p(\mu, \nu) \mid$.

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Semi-discrete OT: fundamental $1/\sqrt{n}$ rate of convergence

Sharper results by investigating $\mathbb{D}(\mu_n, \nu)$ ($\mu \neq \nu$) insted. This is because the sample complexity of OT adapts to the "less complex" measure[12].

[12] Empirical optimal transport between different measures adapts to lower complexity, S.Hundrieser, T.Staudt and A.Munk.2022.

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Sharper results by investigating $\mathbb{D}(\mu_n, \nu)$ $(\mu \neq \nu)$ insted. This is because the sample complexity of OT adapts to the "less complex" measure[12]. In particular if ν is discrete then

$$\mathbb{E}[\mid W_p^p(\mu_n, \nu) - W_p^p(\mu, \nu) \mid] = O(1/\sqrt{n}).$$

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Statistical aspect of barycenters

Empirical Wasserstein barycenters are solutions of $\min_{\mu\in\mathcal{M}_1(\mathbb{R}^d)} \frac{1}{L}\sum_{l=1}^{L} W_p^p(\mu_n^l,\mu)$ with for all $l=1,\dots,L$ μ_n^l empirical measures over i.i.d realizations of μ^l.

[13] Quantitative stability of barycenters in the Wasserstein space, G.Carlier, A.Delalande and Q.Mérigot. 2022.

[14] Randomized Wasserstein barycenter computation: Resampling with statistical guarantees, F.Heinemann, A.Munk and Y.Zemel. 2023.

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$$\mathbb{E}\left[W_2^2\left(\mu_n^*,\mu^*\right)\right] \lesssim \begin{cases} n^{-1/12} \text{ if } d<4 \\ n^{-1/12}\log(n)^{1/6} \text{ if } d=4 \\ n^{-1/3d} \text{ if } d>4. \end{cases}$$

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In the discrete setting we have[14]

$$\mathbb{E}\left[\frac{1}{L}\sum_{l=1}^{L} W_p^p(\mu^l, \nu_n) - \min_{\nu \in \mathcal{M}_1(\mathbb{R}^d)} \frac{1}{L}\sum_{l=1}^{L} W_p^p(\mu^l, \nu)\right] \lesssim \frac{1}{\sqrt{n}}.$$

Where ν_n is an empirical barycenter.

[13] Quantitative stability of barycenters in the Wasserstein space, G.Carlier, A.Delalande and Q.Mérigot. 2022.

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